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Deformed Minkowski spaces: classification and properties

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Abstract

Using general but simple covariance arguments, we classify the 'quantum' Minkowski spaces for dimensionless deformation parameters. This requires a previous analysis of the associated Lorentz groups, which reproduces a previous classification by Woronowicz and Zakrzewski. As a consequence of the unified analysis presented, we give the commutation properties, the deformed (and central) length element and the metric tensor for the different spacetime algebras.

1 Introduction

Following the approach of [1], we present here a classification of the possible deformed Minkowski spaces (algebras). Our analysis, which provides a common framework for the properties of the various Minkowski spacetimes, requires the consideration of the two ($SL_q(2)$ and $SL_h(2)$) deformations of $SL(2,C)$ and provides a characterization of the appropriate R-matrices defining the deformed Lorentz groups given in [2] (see also [3]).

It is well known that $GL(2,C)$ admits only two different deformations having a central determinant: one is the standard q-deformation [4, 5] and the other is the non-standard or 'Jordanian' h-deformation [6, 7, 8]. Both $GL_q(2)$ and $GL_h(2)$ have associated 'quantum spaces' in the sense of [9]. These deformations (which may be shown to be related by contraction [10]) are defined as the associative algebras generated by the entries a, b, c, d of a matrix M, the commutation properties of which may be expressed by an 'FRT' [5] equation

$$R_{12}M_1M_2 = M_2M_1R_{12} \tag{1}$$

for a suitable R-matrix. Let us summarize their properties.

a) For $GL_q(2)$ the R-matrix in (1) is ($\lambda \equiv q - q^{-1}$)

$$R_q = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix} \quad , \quad \hat{R}_q \equiv \mathcal{P} R_q = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{bmatrix} \quad , \quad \mathcal{P} R_q \mathcal{P} = R_q^t \; , \qquad (2)$$

where $\mathcal{P}$ is the permutation operator ($\mathcal{P} = \mathcal{P}^\dagger$, $\mathcal{P}_{ij,kl} = \delta_{il}\delta_{jk}$), and the commutation relations defining the quantum group algebra are

$$\begin{array}{llllll} ab = qba & , & ac = qca & , & ad - da = \lambda bc & , \\ bc = cb & , & bd = qdb & , & cd = qdc & . \end{array} \qquad (3)$$

$Fun(GL_q(2))$ has a quadratic central element,

$$det_q M := ad - qbc \quad ; \qquad (4)$$

$det_q M = 1$ defines $SL_q(2)$. The matrix $\hat{R}_q \equiv \mathcal{P} R_q$ satisfies Hecke's condition

$$\hat{R}_q^2 - \lambda \hat{R}_q - I = 0 \quad , \quad (\hat{R}_q - qI)(\hat{R}_q + q^{-1} I) = 0 \quad , \qquad (5)$$

and (we shall assume $q^2 \neq -1$ [5] throughout) it has a spectral decomposition in terms of a rank three projector P_{q+} and a rank one projector P_{q-},

$$\hat{R}_q = qP_{q+} - q^{-1} P_{q-} \; , \; \hat{R}_q^{-1} = q^{-1} P_{q+} - qP_{q-} \; , \; [\hat{R}_q, P_{q\pm}] = 0 \; , \; P_{q\pm} \hat{R}_q P_{q\mp} = 0 \; , \qquad (6)$$

$$P_{q+} = \frac{I + q\hat{R}_q}{1+q^2} \quad , \quad P_{q-} = \frac{I - q^{-1}\hat{R}_q}{1+q^{-2}} = \frac{1}{1+q^{-2}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-2} & -q^{-1} & 0 \\ 0 & -q^{-1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \; . \qquad (7)$$

The following relations have an obvious equivalent in the undeformed case:

$$\epsilon_q M^t \epsilon_q^{-1} = M^{-1} \; , \quad \epsilon_q = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix} = -\epsilon_q^{-1} \; , \quad P_{q-\; ij,kl} = \frac{-1}{[2]_q} \epsilon_{q\; ij} \epsilon_{q\; kl}^{-1} \; . \qquad (8)$$

The determinant of an ordinary 2×2 matrix may be defined as the proportionality coefficient in $(detM)P_- := P_- M_1 M_2$ where P_- is given by (7) for q=1. In the $q \neq 1$ case the q-determinant (4) may be expressed as

$$(det_q M) P_{q-} := P_{q-} M_1 M_2 \quad , \quad (det_q M^{-1}) P_{q-} = M_2^{-1} M_1^{-1} P_{q-} \quad , \qquad (9)$$

($det_q M^{-1} = (det_q M)^{-1}$, and $(det_q M)^\dagger P_{q-}^\dagger = M_2^\dagger M_1^\dagger P_{q-}^\dagger$).

b) For $GL_h(2)$ the R-matrix in (1) is the solution of the Yang-Baxter equation given by

$$R_h = \begin{bmatrix} 1 & -h & h & h^2 \\ 0 & 1 & 0 & -h \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix} \; , \; \hat{R}_h \equiv \mathcal{P} R_h = \begin{bmatrix} 1 & -h & h & h^2 \\ 0 & 0 & 1 & h \\ 0 & 1 & 0 & -h \\ 0 & 0 & 0 & 1 \end{bmatrix} \; , \; \mathcal{P} R_h \mathcal{P} = R_h^{-1} \; , \qquad (10)$$

(or $R_{h\,12}R_{h\,21} = I$, triangularity condition) for which (1) gives

$$\begin{array}{lll} [a,b] = h(\xi - a^2) \quad , & [a,c] = hc^2 \quad , & [a,d] = hc(d-a) \quad , \\ [b,c] = h(ac+cd) \quad , & [b,d] = h(d^2 - \xi) \quad , & [c,d] = -hc^2 \end{array} \tag{11}$$

(so that $[a-d, c] = 0$ follows), where ξ is the quadratic central element

$$\xi \equiv det_h M = ad - cb - hcd \quad ; \tag{12}$$

setting $\xi = 1$ reduces $GL_h(2)$ to $SL_h(2)$. The matrix $\hat{R}_h$ satisfies

$$\hat{R}_h^2 = I \quad , \quad (I - \hat{R}_h)(I + \hat{R}_h) = 0 \quad . \tag{13}$$

It has two eigenvalues (1 and -1) and a spectral decomposition in terms of a rank three projector P_{h+} and a rank one projector P_{h-}

$$\hat{R}_h = P_{h+} - P_{h-} \quad , \qquad P_{h\pm}\hat{R}_h = \pm P_{h\pm} \quad , \tag{14}$$

$$P_{h+} = \frac{1}{2}(I + \hat{R}_h) \quad , \quad P_{h-} = \frac{1}{2}(I - \hat{R}_h) = \frac{1}{2}\left[\begin{array}{cccc} 0 & h & -h & -h^2 \\ 0 & 1 & -1 & -h \\ 0 & -1 & 1 & h \\ 0 & 0 & 0 & 0 \end{array}\right] \quad . \tag{15}$$

For $SL_h(2)$, the formulae equivalent to those in (8) are

$$\epsilon_h M^t \epsilon_h^{-1} = M^{-1} \; , \; \epsilon_h = \left(\begin{array}{cc} h & 1 \\ -1 & 0 \end{array}\right) \; , \; \epsilon_h^{-1} = \left(\begin{array}{cc} 0 & -1 \\ 1 & h \end{array}\right) \; , \; P_{h-\,ij,kl} = \frac{-1}{2}\epsilon_{h\,ij}\epsilon^{-1}_{h\,kl} \; . \tag{16}$$

Using P_{h-}, the deformed determinant and its inverse, $det_h M$ and $det_h M^{-1}$, (12) are also given by eqs. (9).

The quantum planes [9] associated with $SL_q(2)$ and $SL_h(2)$ are the associative algebras generated by two elements $(x, y) \equiv X$, the commutation properties of which (explicitly and in R-matrix form) are

a) for $SL_q(2)$ [9]

$$xy = qyx \quad \longleftrightarrow \quad R_q X_1 X_2 = q X_2 X_1 \quad , \tag{17}$$

b) for $SL_h(2)$ [7, 8]

$$xy = yx + hy^2 \quad \longleftrightarrow \quad R_h X_1 X_2 = X_2 X_1 \quad . \tag{18}$$

These commutation relations are preserved under transformations by the corresponding quantum groups matrices[1] M, $X' = MX$. This invariance statement, suitably extended to apply to the case of deformed Minkowski spaces, provides the essential ingredient for their classification.

From now on we shall often write R_Q, P_Q ($Q = q, h$) to treat both deformations simultaneously. For instance, (17) and (18) may be jointly written as $R_Q X_1 X_2 = \rho X_2 X_1$, where $\rho = (q, 1)$ is the appropriate eigenvalue of R_Q.

[1] The $GL_q(2)$ and $GL_h(2)$ matrices also preserve, respectively, the 'q-symplectic' and 'h-symplectic' metrics ϵ_q (or $\epsilon_q{}^{-1}$) and $\epsilon_h{}^{-1}$.

2 Deformed Lorentz groups and associated Minkowski algebras

As is well known, the vector representation $D^{\frac{1}{2},\frac{1}{2}} = D^{\frac{1}{2},0} \otimes D^{0,\frac{1}{2}}$ of the restricted Lorentz group may be given by the transformation $K' = AKA^\dagger$, $A \in SL(2,C)$. The spacetime coordinates are contained in $K = K^\dagger = \sigma^\mu x_\mu$, where $\sigma^0 = I$ and σ^i are the Pauli matrices; the time coordinate may be identified as $x^0 = \frac{1}{2} tr(K)$. Since $det K = (x_0)^2 - x^i x_i = det K'$, the correspondence $\pm A \mapsto \Lambda \in SO(1,3)$, where $x'^\mu = \Lambda^\mu_{\ \nu} x^\nu$, realizes the covering homomorphism $SL(2,C)/Z_2 = SO(1,3)$. A first step to obtain a deformation of the Lorentz group is to replace the $SL(2,C)$ matrices A above by the generator matrix M of $SL_q(2)$ [11, 12, 13, 14].

In general, the full determination of a deformed Lorentz group requires the characterization of all possible commutation relations among the generators (a,b,c,d) of M and (a^*,b^*,c^*,d^*) of $M^\dagger$, M being a deformation of $SL(2,C)$. The R-matrix form of these may be expressed in full generality by

$$\begin{array}{llll} R^{(1)} M_1 M_2 = M_2 M_1 R^{(1)} & , & M_1^\dagger R^{(2)} M_2 = M_2 R^{(2)} M_1^\dagger & , \\ M_2^\dagger R^{(3)} M_1 = M_1 R^{(3)} M_2^\dagger & , & R^{(4)} M_1^\dagger M_2^\dagger = M_2^\dagger M_1^\dagger R^{(4)} & , \end{array} \qquad (19)$$

where $R^{(3)\,\dagger} = R^{(2)} = \mathcal{P} R^{(3)} \mathcal{P}$ (or 'reality' condition[2] for $R^{(3)}$) and $R^{(4)} = R^{(1)\,\dagger}$ or $R^{(4)} = (\mathcal{P} R^{(1)\,-1} \mathcal{P})^\dagger$ since the first eq. in (19) is invariant under the exchange $R^{(1)} \leftrightarrow \mathcal{P} R^{(1)\,-1} \mathcal{P}$.

Eqs. (19), which also follow (see *e.g.* [15]) from the bi-spinor (dotted and undotted) description of 'quantum' spacetime in terms of a deformed K, will be taken as the starting point for the classification of the deformed Lorentz groups. In it, the matrix $R^{(1)}$ characterizes the appropriate deformation of the $SL(2,C)$ group ($R^{(1)} = R_Q$), $R^{(2)}$ (or $R^{(3)}$) defines how the elements of M and $M^\dagger$ commute and it is not *a priori* fixed (but it must satisfy consistency relations with $R^{(1)}$, see eq. (20) below) and $R^{(4)}$ gives the commutation relations for the complex conjugated generators contained in $M^\dagger$. The specification of the deformed Lorentz group will be completed by the commutation properties of the generators with their complex conjugated ones *i.e.*, by the determination of $R^{(2)} = R^{(3)\,\dagger}$.

The commutation relations of the deformed Lorentz group algebra generators (entries of M and $M^\dagger$) are given by eqs. (19). The consistency of these relations is assured if $R^{(1)}$ (and $R^{(4)}$) obey the Yang-Baxter equation (YBE) and $R^{(3)}$ and $R^{(2)}$ satisfy the mixed consistency equations [1, 16]

$$R_{12}^{(1)} R_{13}^{(3)} R_{23}^{(3)} = R_{23}^{(3)} R_{13}^{(3)} R_{12}^{(1)} \quad , \quad R_{12}^{(4)} R_{13}^{(2)} R_{23}^{(2)} = R_{23}^{(2)} R_{13}^{(2)} R_{12}^{(4)} \quad , \qquad (20)$$

(these two equations are actually the same since either $R^{(4)} = R^{(1)\,\dagger}$ or $R^{(4)} = (\mathcal{P} R^{(1)\,-1} \mathcal{P})^\dagger$ and $R^{(2)} = R^{(3)\,\dagger}$). It will be convenient to notice that the first

[2] This reality condition can be given in a more general form $R^{(3)\,\dagger} = \tau \mathcal{P} R^{(3)} \mathcal{P}$ for $|\tau| = 1$, however this phase factor can be eliminated by the redefinition $R^{(3)} \to \tau^{1/2} R^{(3)}$ (cf. [2]).

equation, considered as an 'RTT' equation, indicates that $R^{(3)}$ is a representation of the deformed $GL(2,C)$ group, *i.e.*, the matrix $R^{(3)}$ provides a 2×2 representation of the entries M_{ij} of the generator matrix M: $(M_{ij})_{\alpha\beta} = R^{(3)}_{i\alpha,j\beta}$. Thus, $R^{(3)}$ may be seen as a matrix in which the 2×2 blocks satisfy among themselves the same commutation relations that the entries of M,

$$R^{(3)} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \sim M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} ,$$

and the problem of finding all possible Lorentz deformations is equivalent to finding all possible $R^{(3)}$ matrices with 2×2 block entries satisfying (3) or (11) such that $\mathcal{P}R^{(3)}\mathcal{P} = R^{(3)\dagger}$ ($\hat{R}^{(3)} = \hat{R}^{(3)\dagger}$).

To introduce the deformed Minkowski *algebra* $\mathcal{M}^{(j)}$ associated with a deformed Lorentz group $L^{(j)}$ (where the index j refers to the different cases) it is natural to extend $K' = AKA^\dagger$ above to the deformed case by stating that in it the corresponding K generates a comodule algebra for the coaction ϕ defined by

$$\phi : K \longmapsto K' = MKM^\dagger \ , \ K'_{is} = M_{ij}M^\dagger_{ls}K_{jl} \ , \ K = K^\dagger \ , \ \Lambda = M \otimes M^* \ , \quad (21)$$

where it is assumed that the matrix elements of K, which now do not commute among themselves, commute with those of M and $M^\dagger$. As in (17), (18) for q-two-vectors (rather, two-*spinors*) we now demand that the commuting properties of the entries of K are preserved by (21). The use of covariance arguments to characterize the algebra generated by the entries of K has been extensively used, and the resulting equations are associated with the name of reflection equations [17, 18] or, in a more general setting, braided algebras [19, 20] of which the former constitute the 'algebraic sector' (for an introduction to braided geometry see [21]); similar equations were also early introduced in [16]. Let us now extend the arguments given in [1] to classify the deformed Lorentz groups and their associated Minkowski algebras in an unified way.

This is achieved by describing the commutation properties of the entries of the hermitian matrix K generating a possible Minkowski algebra $\mathcal{M}$ by means of a general reflection equation of the form

$$R^{(1)}K_1R^{(2)}K_2 = K_2R^{(3)}K_1R^{(4)} \quad , \qquad (22)$$

where the $R^{(i)}$ matrices ($i = 1, ..., 4$) are those introduced in (19). Indeed, writing equation (22) for $K' = MKM^\dagger$, it follows that the invariance of the commutation properties of K under the associated deformed Lorentz transformation (21) is achieved if relations (19) are satisfied.

The deformed Minkowski length and metric, invariant under a Lorentz transformation (21) of $L^{(j)}$, is defined through the quantum determinant of K. Since the two matrices $\hat{R}^{(1)} = \mathcal{P}R_Q$ have spectral decompositions ((6), (14)) with a rank three projector P_{Q+} and a rank one projector P_{Q-}, and

the determinants of M, $M^\dagger$ are central (eqs. (9), (12)), the Q-*deformed* and *invariant* (under (21)) determinant of the $2{\times}2$ matrix K may now be given by

$$(det_Q K) P_{Q-} P^\dagger_{Q-} = -\rho P_{Q-} K_1 \hat{R}^{(3)} K_1 P^\dagger_{Q-} \quad . \tag{23}$$

It is easy to check that $(P_{Q-}P^\dagger_{Q-})^2 = \left(\frac{\omega_Q}{|[2]_\rho|}\right)^2 P_{Q-}P^\dagger_{Q-}$, where $\omega_q = |q| + |q^{-1}|$, $\omega_h = 2 + h^2$ and $[2]_1 = 2$. In eq. (23), the subindex Q in $det_Q K$ indicates that it depends on q or h (or on other parameters on which $R^{(3)}$ may depend) and ρ ($= (q,1)$ as before) has been added by convenience. Since $\hat{R}^{(3)}$ and K are hermitian, $det_Q K$ is real (if ρ is not real it may be factored out). We stress that the above formula provides a general expression for a central (see below) quadratic element which constitutes the *deformed Minkowski length* for *all* deformed spacetimes $\mathcal{M}^{(j)}$.

Similarly, it is possible to write in general the invariant scalar product of *contravariant* (transforming as the matrix K, eq. (21)) and *covariant* (transforming by $Y \mapsto Y' = (M^\dagger)^{-1} Y M^\dagger$) matrices (four-vectors) as the quantum trace of a matrix product [1] (cf. [5]). In the present general case, the deformed trace of a matrix B is defined by

$$tr_Q(B) := tr(\mathcal{D}_Q B) \quad , \quad \mathcal{D}_Q = \rho^2 tr_{(2)}(\mathcal{P}(((R_Q)^{t_1})^{-1})^{t_1}) \quad , \tag{24}$$

where $tr_{(2)}$ means trace in the second space. This deformed trace is invariant under the quantum group coaction $B \mapsto MBM^{-1}$ since the expression of $\mathcal{D}_Q$ above guarantees that $\mathcal{D}^t_Q = M^t \mathcal{D}^t_Q (M^{-1})^t$ is fulfilled. In particular, the $\mathcal{D}_Q$ matrices for R_q and R_h are found to be

$$\mathcal{D}_q = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} \quad , \quad \mathcal{D}_h = \begin{pmatrix} 1 & -2h \\ 0 & 1 \end{pmatrix} \quad . \tag{25}$$

Let us now find the expression of the metric tensor. Using ϵ_Q (cf. (8), (16)) $(P_{Q-})_{ij,kl} = -\frac{1}{[2]_\rho} \epsilon_{Q\,ij} \epsilon^{-1}_{Q\,kl}$ and $\mathcal{D}_Q = -\epsilon_Q (\epsilon^{-1}_Q)^t$ ($\mathcal{D}^t_Q = M^t \mathcal{D}^t_Q (M^{-1})^t$ now follows from $\epsilon_Q M^t \epsilon^{-1}_Q = M^{-1}$, eqs. (8) and (16)). The covariant K^ϵ_{ij} vector is

$$K^\epsilon_{ij} = \hat{R}^\epsilon_{Q\,ij,kl} K_{kl} \quad , \quad \hat{R}^\epsilon_Q \equiv (1 \otimes (\epsilon^{-1}_Q)^t) \hat{R}^{(3)} (1 \otimes (\epsilon^{-1}_Q)^\dagger) \quad , \tag{26}$$

from which follows that the general Minkowski length and metric is given by

$$l_Q \equiv det_Q K = \frac{\rho}{\omega_Q} tr_Q K K^\epsilon \equiv \rho^2 g_{Q\,ij,kl} K_{ij} K_{kl} \quad , \quad g_{Q\,ij,kl} = \frac{\rho^{-1}}{\omega_Q} \mathcal{D}_{Q\,si} \hat{R}^\epsilon_{Q\,js,kl} \,. \tag{27}$$

This concludes the unified description of all cases. Let us now look at their classification and specific properties.

3 Characterization of the Lorentz deformations

First we use the reality condition $R^{(3)\,\dagger} = \mathcal{P}R^{(3)}\mathcal{P}$ to reduce the number of independent parameters in $R^{(3)}$. It implies

$$R^{(3)} \equiv \begin{bmatrix} A & B \\ C & D \end{bmatrix} \equiv \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ c_{11} & c_{12} & d_{11} & d_{12} \\ c_{21} & c_{22} & d_{21} & d_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{21}^* & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ a_{12}^* & c_{12} & a_{22}^* & c_{22}^* \\ b_{12}^* & c_{22} & b_{22}^* & d_{22} \end{bmatrix} , \tag{28}$$

where a_{11}, d_{22}, b_{21}, c_{12} are real numbers and the rest are complex.

a) Deformed Lorentz groups associated with $SL_q(2)$

Let now $M \in SL_q(2)$ and $R^{(1)} = R_q$, eq. (2). The problem of finding the q-Lorentz groups associated with the standard deformation is now reduced to obtaining all matrices $R^{(3)}$ satisfying (20). This means that the 2×2 matrices A, B, C, D in (28) must satisfy the commutation relations in (3). This implies that (see [2]) $B^2 = C^2 = 0$, $AD \sim I_2$ and that either B or C are zero. Now

a1) $B = 0$ gives $R^{(3)} = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & c_{12} & a_{22}^* & 0 \\ 0 & 0 & 0 & d_{22} \end{bmatrix}$ with a_{11}, d_{22}, $c_{12} \in R$.

From $AD \sim I_2$ it is easy to see (fixing first $a_{11} = 1$) that $d_{22} = a_{22}^*/a_{22}$; its reality then implies $d_{22} = \pm 1$, $d_{22} = 1$ when $a_{22} \in R$ and $d_{22} = -1$ for $a_{22} \in iR$. The relation $AC = qCA$ forces $a_{22} = q^{-1}$ or $c_{12} = 0$.

a2) $C = 0$ gives $R^{(3)} = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & b_{21} & 0 \\ 0 & 0 & a_{22}^* & 0 \\ 0 & 0 & 0 & d_{22} \end{bmatrix}$ with a_{11}, d_{22}, $b_{21} \in R$, $a_{11} = 1$;

as in the previous case, $d_{22} = \pm 1$ and $a_{22} \in R$ for $d_{22} = 1$ and $a_{22} \in iR$ for $d_{22} = -1$. Analogously, from $AB = qBA$ one obtains that $b_{21} = 0$ or $a_{22} = q$.

Thus, the solutions for $R^{(3)}$ are the following

$$R^{(3)} = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & r & 1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix} , \quad \begin{array}{l} q \in R , \\ r \in R , \end{array} \tag{29}$$

$$R^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & \pm t & 0 \\ 0 & 0 & 0 & \pm 1 \end{bmatrix} , \quad \begin{array}{l} + \text{ for } t \in R , \\ - \text{ for } t \in iR , \end{array} \tag{30}$$

$$R^{(3)} = \begin{bmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & r & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{bmatrix} \quad , \quad \begin{array}{l} q \in R\ , \\ r \in R\ , \end{array} \tag{31}$$

$$R^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & r & -q^{-1} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad , \quad \begin{array}{l} q \in iR\ , \\ r \in R\ , \end{array} \tag{32}$$

$$R^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & q & r & 0 \\ 0 & 0 & -q & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad , \quad \begin{array}{l} q \in iR\ , \\ r \in R\ , \end{array} \tag{33}$$

Remarks:

- Notice that, as anticipated, the Q-'determinant' of all these $R^{(3)}$ matrices, computed as $det_Q M$, is a scalar -hence commuting- 2×2 matrix.
- $R_q^\dagger = \mathcal{P} R_q \mathcal{P}$ iff $q \in R$. Hence, $R_{12}^{(4)} = R_{q21}$ or R_{q12}^{-1}. Thus $\tilde{M} \equiv (M^{-1})^\dagger$ provides a second copy of $SL_q(2)$, since then $R_q \tilde{M}_1 \tilde{M}_2 = \tilde{M}_2 \tilde{M}_1 R_q$.
- The case (29) for $r = q - q^{-1} = \lambda$ ($R^{(3)} = R_q$) is the quantum Lorentz group of [12, 13] ($L_q^{(1)}$ in the notation of [1]). If $r \neq \lambda$ we obtain a 'gauged' version of it: $R^{(3)} = e^{\alpha\sigma_2^3} R_q e^{-\alpha\sigma_2^3}$ ($r = \lambda e^{2\alpha}$), where the subindex in σ_2^3 refers to the second space.
- The matrix (30) for $t = 1$ and $q \in R$ corresponds to $L_q^{(2)}$ in [1].
- The calculations leading to (30)-(33) require assuming $q^2 \neq 1$. However, the solutions for $q \in R$ are also valid in the limit $q = 1$ (see [2]); in this limit ($R^{(1)} = R^{(4)} = I_4$), the case (30) gives the deformed Lorentz group (twisted) of [22]. For $q = -1$, additional solutions appear and, although we shall not discuss these particular cases (see [2]), the associated Minkowski algebras may be obtained as in the general q case.
- These results coincide with the classification in [2]: the solutions (30) correspond to eqs. (13) and (14) in [2]; similarly, (29), (31), (32) and (33) correspond to (74) (q real), (15), (74) (q imaginary) and (16) in that reference.

b) Deformed Lorentz groups associated with $SL_h(2)$

Let now $R^{(1)} = R_h$, eq. (10). For h imaginary, $h \in iR$, the matrix R_h satisfies the reality condition $R_h^* = R_h^{-1}$ ($= \mathcal{P} R_h \mathcal{P}$); this means that $\tilde{M} \equiv M^*$ defines a second copy of $SL_h(2)$ since $R_h M_1^* M_2^* = M_2^* M_1^* R_h$. The value of $h \in C\backslash\{0\}$, however, is not important. Indeed, quantum groups related with two different values of $h \in C$ are equivalent and their R matrices are related by a similarity transformation[3]; thus, we can take $h \in R$ or even $h = 1$.

Since the entries of M satisfy (11), the 2×2 blocks in $R^{(3)}$ (eq. (28)) will satisfy now these commutation relations. This leads to (see [2]) $C = 0$ so that,

[3]Quantum groups associated with R_h and $R_{h=1}$ are related by a similarity transformation defined by the 2×2 matrix $S = diag(h^{-1/2}, h^{1/2})$: $R_{h=1} = (S \otimes S) R_h (S \otimes S)^{-1}$.

taking the h-'determinant' of $R^{(3)}$ equal I_2, the set of commutation relations reduces to

$$AD = I_2 \quad , \qquad [A, B] = h(I_2 - A^2) \quad . \tag{34}$$

Using them in (28) the following solutions for $R^{(3)}$ are found ($h \in R$)

$$R^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & r & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad , \quad r \in R \quad , \tag{35}$$

$$R^{(3)} = \begin{bmatrix} 1 & 0 & -h & 0 \\ -h & 1 & r & h \\ 0 & 0 & 1 & 0 \\ 0 & 0 & h & 1 \end{bmatrix} \quad , \quad \begin{array}{l} h \in R \, , \\ r \in R \, . \end{array} \tag{36}$$

Remarks:

- In (36), for $r = h^2$ we have $R^{(3)} = (\mathcal{P} R_h \mathcal{P})^{t_2}$. However, the parameter r can be removed with an appropriate change of basis provided $h \neq 0$. For $h = 0$, this is not possible and constitutes a different case, eq. (35). This case is another example where the non-commutativity is solely due to $R^{(3)} \neq I_4$.
- The cases (35), (36) correspond to (20) and (21) [cf. (78)] in [2].

4 Minkowski algebras: classification and properties

We now present here, in explicit form, the commutation relations for the generators of the deformed Minkowski spacetimes; they follow easily from (22) using the previous $R^{(3)}$ matrices. We saw in (19) that $R^{(3)\,\dagger} = R^{(2)} = \mathcal{P} R^{(3)} \mathcal{P}$ and $R^{(4)} = R^{(1)\,\dagger}$ or $R^{(4)} = (\mathcal{P} R^{(1)\,-1} \mathcal{P})^\dagger$ (these two possibilities are the same for $Q = h$). Clearly, eq. (22) allows for a factor in one side without impairing its invariance properties. This factor may be selected with the (natural) condition that the resulting Minkowski algebra does not contain generators $\alpha, \beta, ...$, with the Grassmann-like property $\alpha^2 = \beta^2 = ... = 0$. In terms of P_{Q+}, this tantamount to requiring that $P_{Q+} K_1 \hat{R}^{(3)} K_1 P_{Q+}^\dagger$ must be non-zero. This leads to (cf. (22)) the equations

$$R_Q K_1 R^{(2)} K_2 = \pm K_2 R^{(3)} K_1 R_Q^\dagger \quad (+ \text{ for } q, h \in R \, , \ - \text{ for } q \in iR) \, . \tag{37}$$

In the q-case we might also consider $R^{(4)} = (\mathcal{P} R^{(1)\,-1} \mathcal{P})^\dagger$. However using Hecke's condition for $R^{(1)}$ it is seen that this leads to the same algebra as (37) with the restriction $det_q K = 0$, so that this case may be considered as included in the previous one.

An important ingredient is the centrality of the Q-determinant (23), $(det_Q K) K = K (det_Q K)$, since it will correspond to the Minkowski length. Using twice (37) we find the following commutation property for three K matrices

$$R_{Q\,13} R_{Q\,23} K_1 R^{(2)}_{12} K_2 R^{(2)}_{13} R^{(2)}_{23} K_3 = K_3 R^{(3)}_{13} R^{(3)}_{23} K_1 R^{(2)}_{12} K_2 R^\dagger_{Q\,13} R^\dagger_{Q\,23} \, . \tag{38}$$

Multiplying from the right by $\mathcal{P}_{12}P^{\dagger}_{Q-\,12}$ and by $P_{Q-\,12}$ from the left and using that R_Q and $R^{(3)}$ represent $GL_Q(2)$ and hence have a central Q-'determinant' represented by a scalar 2×2 matrix we get

$$(det_Q R_Q)(det_Q R^{(3)})^{\dagger}\,(det_Q K)K = (det_Q R_Q)^{\dagger}(det_Q R^{(3)})\,K(det_Q K) \quad , \tag{39}$$

The scalar $det_Q R^{(i)}$ matrices always cancel out in the cases below ($det_q R_q = qI_2$ and $det_h R_h = I_2$) assuring the centrality of $det_Q K$ (as it may be checked by direct computation).

a) q-Minkowski spaces associated with $SL_q(2)$

1) Let us consider the case (29) for $r = \lambda$ (*i.e.*, $R^{(3)} = R_q$, q real). The commutation relations for the entries of $K=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ are

$$\begin{array}{llll} \alpha\beta = q^{-2}\beta\alpha \quad , & [\delta,\beta] = q^{-1}\lambda\alpha\beta \quad , & \alpha\gamma = q^2\gamma\alpha \quad , \\ [\beta,\gamma] = q^{-1}\lambda(\delta-\alpha)\alpha \quad , & [\alpha,\delta] = 0 \quad , & [\gamma,\delta] = q^{-1}\lambda\gamma\alpha \quad ; \end{array} \tag{40}$$

they characterize the algebra $\mathcal{M}_q^{(1)}$ ([12]-[14]; see also [20, 23, 24, 1]). The Minkowski length is given by (23),

$$det_q K = \alpha\delta - q^2\gamma\beta \quad . \tag{41}$$

If $r \neq \lambda$, the commutation relations are slightly different; this, however, corresponds only to an appropriate election of the basis ('gauged' version of this Minkowski space).

2) Let $R^{(3)}$ be given by eq. (30). The centrality of the q-determinant implies that q and t are both real or both imaginary. The commutation relation for the entries of K and the q-Minkowski length (eq. (23)) are (the sign $+$ is for $q,t \in R$ and the $-$ for $q,t \in iR$)

$$\begin{array}{lll} q\alpha\beta = \pm t\beta\alpha \quad , & t\alpha\gamma = \pm q\gamma\alpha \quad , & \alpha\delta = \delta\alpha \quad , \\ [\beta,\gamma] = \pm t\lambda\alpha\delta \quad , & \beta\delta = \pm qt\delta\beta \quad , & \delta\gamma = \pm qt\gamma\delta \quad ; \end{array} \tag{42}$$

$$det_{q,t} K = \frac{q+q^{-1}}{q \pm q^{-1}}(-q\gamma\beta \pm t\alpha\delta) \quad . \tag{43}$$

Remarks:

- For $t = 1$, these commutation relations correspond to the Minkowski algebra $\mathcal{M}_q^{(2)}$ [12, 25, 1] which is isomorphic to the quantum algebra[4] $GL_q(2)$.
- For $q = 1$ and t real, we get the Minkowski space obtained in [22] (denoted $\mathcal{M}^{(3)}$ in [1]). This algebra and the corresponding deformed Poincaré algebra have been shown to be [27] a simple transformation (twisting) of the classical one. As a result, it is possible to remove the non-commuting character of the entries of K [28].

[4]The Minkowski space of [26] is also a $GL_q(2)$-like space, but different from the above.

3) Let us take $R^{(3)}$ as in eq. (31) for $r{=}{-}\lambda$ ($R^{(3)} = \mathcal{P}R_q^{-1}\mathcal{P}$). Then,

$$\begin{array}{lll} [\alpha,\beta] = q\lambda\beta\delta \quad , & [\alpha,\gamma] = -q\lambda\delta\gamma \quad , & [\alpha,\delta] = 0 \quad , \\ [\beta,\gamma] = q\lambda(\alpha-\delta)\delta \quad , & \beta\delta = q^2\delta\beta \quad , & \gamma\delta = q^{-2}\delta\gamma \quad ; \end{array} \tag{44}$$

$$det_q K = q^2\alpha\delta - \beta\gamma \quad . \tag{45}$$

This algebra may also be identified with the algebra of spacetime derivatives in [14] (see also [23]).

4) Let $R^{(3)}$ be now given by (32). The Minkowski algebra and the central length are given by

$$\begin{array}{lll} \alpha\beta = -q^{-2}\beta\alpha \quad , & \delta\beta + \beta\delta = r\alpha\beta \quad , & \alpha\gamma = -q^2\gamma\alpha \quad , \\ [\beta,\gamma] = -q^{-1}\lambda\delta\alpha + r\alpha^2 \quad , & [\alpha,\delta] = 0 \quad , & \gamma\delta + \delta\gamma = r\gamma\alpha \quad , \end{array} \tag{46}$$

$$det_q K = \frac{-q[2]}{\lambda}(q^{-2}\alpha\delta + \gamma\beta) \quad . \tag{47}$$

5) Finally, let $R^{(3)}$ be as in eq. (33). Then,

$$\begin{array}{lll} \alpha\beta + \beta\alpha = -r\beta\delta \quad , & \alpha\gamma + \gamma\alpha = -r\delta\gamma \quad , & [\alpha,\delta] = 0 \quad , \\ [\beta,\gamma] = -q\lambda\alpha\delta + r\delta^2 \quad , & \beta\delta = -q^2\delta\beta \quad , & \gamma\delta = -q^{-2}\delta\gamma \quad ; \end{array} \tag{48}$$

$$det_q K = \frac{-q[2]}{\lambda}(q^2\alpha\delta + \beta\gamma) \quad . \tag{49}$$

b) Deformed Minkowski spaces associated with $SL_h(2)$

1) Let $R^{(3)}$ be given first by eq. (35) and let $R^{(1)} = R_h$, eq. (10). Using (37) with the plus sign and (23) we find (h real)

$$\begin{array}{ll} [\alpha,\beta] = -h\beta^2 - r\beta\delta + h\delta\alpha - h\beta\gamma + h^2\delta\gamma \; , & [\alpha,\delta] = h(\delta\gamma - \beta\delta) \; , \\ [\alpha,\gamma] = h\gamma^2 + r\delta\gamma - h\alpha\delta + h\beta\gamma - h^2\beta\delta \; , & [\beta,\delta] = h\delta^2 \; , \\ [\beta,\gamma] = h\delta(\gamma+\beta) + r\delta^2 \; , & [\gamma,\delta] = -h\delta^2 \; ; \end{array} \tag{50}$$

$$det_h K = \frac{2}{h^2+2}(\alpha\delta - \beta\gamma + h\beta\delta) \quad . \tag{51}$$

2) Let $R^{(3)}$ be given now by eq. (36) with $r = 0$. In this case,

$$\begin{array}{ll} [\alpha,\beta] = 2h\alpha\delta + h^2\beta\delta \quad , & [\alpha,\delta] = 2h(\delta\gamma - \beta\delta) \quad , \\ [\alpha,\gamma] = -h^2\delta\gamma - 2h\delta\alpha \quad , & [\beta,\delta] = 2h\delta^2 \quad , \\ [\beta,\gamma] = 3h^2\delta^2 \quad , & [\gamma,\delta] = -2h\delta^2 \quad ; \end{array} \tag{52}$$

$$det_h K = \frac{2}{h^2+2}(\alpha\delta - \beta\gamma + 2h\beta\delta) \quad . \tag{53}$$

c) Final remarks

For all the Q-spacetime algebras, time may be defined as proportional to $tr_Q K$ ($= 2x^0$ in the undeformed case). The time generator obtained in this way is central only for $\mathcal{M}_q^{(1)}$ [12]-[14] and for the Minkowski algebra (44) (in fact, they are isomorphic: the entries of the covariant vector K^ϵ for $\mathcal{M}_q^{(1)}$ satisfy the commutation relations (44) [1]).

The differential calculus on all the above Minkowski spaces may be easily discussed now along the lines of [1, 23]; one could also investigate the rôle played in it by the contraction relating [10] the q- and h-deformations. To conclude, let us mention that the additive braided group structure [19]-[21] of all these algebras, may be easily found. It suffices to impose that eq. (37) is also satisfied by the sum $K' + K$ of two copies K and K'. Using Hecke's condition ($R_{Q12} = R_{Q21}^{-1} + (\rho - \rho^{-1})\mathcal{P}$) this gives

$$R_Q K_1' R^{(2)} K_2 = \pm K_2 R^{(3)} K_1' (\mathcal{P} R_Q^\dagger \mathcal{P})^{-1} \quad (+ \text{ for } q, h \in R\ ,\ - \text{ for } q \in iR)\ , \tag{54}$$

which is clearly preserved by (21); for $\mathcal{M}_q^{(1)}$, it reproduces the result of [24].

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